

# HOLOGRAPHIC DUALS OF 4D FIELD THEORIES<sup>1</sup>

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## Abstract

We discuss various aspect of the holographic correspondence between 5-d gravity and 4-d field theory. First of all, we describe deformations of  $\mathcal{N} = 4$  Super Yang-Mills (SYM) theories in terms of 5-d gauged supergravity. In particular, we describe  $\mathcal{N} = 0$  and  $\mathcal{N} = 1$  deformations of  $\mathcal{N} = 4$  SYM to confining theories. Secondly, we describe recent proposals for the holographic dual of the renormalization group and for 4-d central charges associated to it. We conclude with a “holographic” proof of the Goldstone theorem.

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The ADS/CFT duality [1, 2, 3] and its extension to non-conformal theories (see [4] for a comprehensive review of the subject) has emerged over the last two years as a powerful tool for understanding strongly coupled field theories. The best studied duality is that between  $\mathcal{N} = 4$  SYM theory with gauge group  $SU(N)$  and coupling constant  $g_{YM}$ , and type IIB superstrings on  $AdS_5 \times S_5$  in the limit

$$N \rightarrow \infty, \quad \lambda = g_S N = \text{constant}. \quad (1)$$

In this duality, when the 't Hooft coupling  $g_S N = g_{YM}^2 N$  is large, the curvature of  $AdS_5 \times S_5$  is small ( $\sim (g_S N)^{-1/4}$ ) so that  $\alpha'$  corrections to type IIB supergravity, i.e. to the low-energy effective action of type IIB superstring, are small. In the large  $N$  limit the string coupling  $g_S$  is vanishingly small; therefore, string loop corrections are also small. This ensures that semiclassical type IIB supergravity is a reliable approximation precisely when perturbative field theory fails.

Gauge-invariant operators of 4-d SYM are related to 10-d fields of type IIB superstring on  $AdS_5 \times S_5$  (see [2, 3] for more details). Since  $S_5$  is compact one can expand the 10-d fields in 5-d KK states propagating on  $AdS_5$ . To identify a 5-d KK mode with a 4-d operator they must transform identically under  $SU(4)$ .  $SU(4)$  is both the isometry group of  $S_5$  (with spinors) and the R-symmetry group of 4-d SYM. The conformal dimension  $\Delta$  of a 4-d operator is determined by the mass of the corresponding 5-d KK mode. For a scalar, it is the largest root of the equation [2]

$$\Delta(\Delta - 4) = (ML)^2; \quad L = (4\pi g_S N \alpha'^2)^{1/4}. \quad (2)$$

Since  $L$  is the  $AdS_5$  radius, a further simplification occurs in the  $g_S N \rightarrow \infty$  limit. The mass of all excited string states is  $O(\alpha'^{-1/2})$  so that their dimension  $\Delta$  diverges in the limit and they decouple from the CFT. Only the conformal dimension of states with mass  $O(1/L)$  remains finite in the limit. These are precisely the KK modes of the states of 10-d type IIB *supergravity* (i.e. the states belonging to the 10-d graviton supermultiplet).

This result fits nicely with superconformal field theory expectations, since (only) the KK states fit into short multiplets of the  $\mathcal{N} = 4$  superconformal algebra and consequently their conformal dimension is protected by a non-renormalization theorem.

Among the scalar operators belonging to short multiplets, 42 are particularly interesting. They are associated to 5-d KK modes that survive a dimensional reduction to 5-d. In other words, 10-d type IIB supergravity on  $AdS_5 \times S_5$  can be consistently truncated to a 5-d supergravity on  $AdS_5$  that contains only these 42 fields, together with their partners under 5-d  $\mathcal{N} = 8$  supersymmetry. No other truncation to a finite subset of KK modes exists.

These KK modes correspond to 4-d composite operators in the SYM theory [3, 5, 6].

The  $\mathcal{N} = 4$  supermultiplet in 4-d contains one vector  $A_\mu$ , four spin-1/2 fermions  $\lambda^I$  and six scalars  $\phi^A$ , all in the adjoint of the gauge group ( $SU(N)$  in our case).  $I$  labels the  $\underline{4}$  and  $A$  labels the  $\underline{6}$  of the R-symmetry group  $SU(4)$ . Under R-symmetry, the 42 operators decompose into a real  $\underline{20}$ , of conformal dimension 2, a complex  $\underline{10}$ , of conformal dimension 3, and a complex singlet of conformal dimension 4. The dimension-2 operators are symmetric, traceless tensors of  $SO(6) \sim SU(4)$ :  $\text{Tr } \phi^{(A} \phi^{B)T}$ . The dimension-3 operators are made of a fermion mass terms plus scalar trilinears, both symmetric tensors of  $SU(4)$ :  $\text{Tr } \lambda^I \lambda^J + O(\phi^3)$ . The dimension-4 operator is simply the  $\mathcal{N} = 4$  Lagrangian plus theta-term.

A deformation of  $\mathcal{N} = 4$  SYM by the 42 operators just discussed above can be described using the dimensional reduction of 10-d type IIB supergravity to 5-d gauged supergravity [7]. This theory has a complicated potential with several stationary points, besides the  $SU(4)$ -invariant one. The holographic correspondence suggests that each (stable) stationary point of the potential describes a conformal field theory [6, 8]. A relevant deformation of  $\mathcal{N} = 4$  SYM generates a flow to another –possibly trivial– local CFT. The holographic equivalent of this RG flow is an appropriate solution of the equations of motion of 5-d supergravity. Since we do not want to break Poincaré invariance, the ansatz for the 5-d metric is

$$ds^2 = dy^2 + e^{2\phi(y)} dx^\mu dx_\mu, \quad \mu = 0, 1, 2, 3. \quad (3)$$

The coordinate  $y$  plays the role of RG scale [6], with larger  $y$  corresponding to higher energy. The background corresponding to a conformal field theory is an  $AdS_5$  metric, with all 42 scalars  $\lambda^a$  at a stationary point of the potential.

The only nonzero fields in our background are the metric and the scalars, so that the relevant part of the 5-dimensional supergravity action is

$$L = \sqrt{-g} \left[ \frac{R}{4} + \frac{1}{2} \sum_a (\partial_y \lambda^a)^2 + V(\lambda^a) \right]. \quad (4)$$

Here we have chosen for simplicity and without loss of generality canonical kinetic term for all scalars. Einstein's equations and the equations of motion of the scalars following from eq. (4) are:

$$\partial_y^2 \lambda^a + 4 \partial_y \phi \partial_y \lambda^a = \frac{\partial V}{\partial \lambda^a}, \quad 6(\partial_y \phi)^2 = \sum_a (\partial_y \lambda^a)^2 - 2V. \quad (5)$$

Eqs. (5) have solutions interpolating between two  $AdS_5$  regions [6], as well as a universal runaway solution, independent of the detailed form of the potential [10].

In both cases for  $y \rightarrow +\infty$  the solution asymptotes to the  $SU(4)$ -invariant,  $\mathcal{N} = 4$  stationary point:

$$\lim_{y \rightarrow +\infty} \lambda^a(y) = 0, \quad \lim_{y \rightarrow +\infty} \phi(y)/y = 1/L. \quad (6)$$

In the solutions of ref. [6], the metric is  $AdS_5$  also in the limit  $y \rightarrow -\infty$  (with a different *larger* curvature). Those solutions are probably pathological because their  $y = -\infty$  (IR) stationary points are non-supersymmetric and unstable [9].

Ref. [10] exhibited a different solution, singular in the IR. In that solution the scalars and the Einstein metric are complicated and non-universal, *but their infrared behavior is universal*. As shown in [10], whenever the metric and scalars become singular at  $y = a$ , and whenever the scalar kinetic term is more singular than the potential, one finds that the metric eq. (3) has the following universal behavior:

$$ds^2 = dy^2 + |y - a|^{1/2} dx^\mu dx_\mu. \quad (7)$$

Eq. (7) agrees with the near-singularity form of the 5-d Einstein-frame metric found in refs. [11, 13, 14]. As the example in [11] shows, the singularity is sometimes an artifact of the 5-d Einstein frame.

The singular metric described here is dual to a deformation of  $\mathcal{N} = 4$  SYM to a confining theory. Confinement can be proven by studying the Wilson loop using the technique of [15] as shown in [10].

A deformation that preserves  $\mathcal{N} = 1$  supersymmetry is non generic, thus, the 5-d metric that gives a holographic description of the deformation is not of the form given in eq. (7). In terms of  $\mathcal{N} = 1$  superfields,  $\mathcal{N} = 4$  contains a vector superfield  $V$  and three chiral superfields,  $\Phi^i$ ,  $i = 1, 2, 3$ , that transform in the  $\underline{3}$  of  $SU(3)$ .  $SU(3)$  is the subgroup of the  $\mathcal{N} = 4$  R-symmetry  $SU(4)$  that commutes with the  $\mathcal{N} = 1$  supercharge. We can deform  $\mathcal{N} = 4$  SYM to pure  $\mathcal{N} = 1$  SYM by adding the  $\mathcal{N} = 1$  supersymmetric F-term  $m \int d^2\theta \text{Tr } \Phi^i \Phi^i$  to the  $\mathcal{N} = 4$  Lagrangian.

The 5-d field corresponding to this deformation is uniquely identified [16] by first decomposing the  $\underline{10}$  of  $SU(4)$  under  $SU(3)$ :

$$\underline{10} \rightarrow \underline{1} + \underline{3} + \underline{6}, \quad (8)$$

and by further decomposing the  $\underline{6}$  of  $SU(3)$  as  $\underline{1} + \underline{5}$  under  $SO(3) \subset SU(3)$ .

As shown in ref. [17], a background of 5-d supergravity that preserves  $\mathcal{N} = 1$  supersymmetry exists if the scalar potential  $V$  can be written in terms of a superpotential  $W$  as

$$V = \frac{1}{8} \sum_{a=1}^n \left| \frac{\partial W}{\partial \lambda^a} \right|^2 - \frac{1}{3} |W|^2, \quad (9)$$

and if the fields satisfy the first-order equation

$$\dot{\lambda}^a = \frac{1}{2} \frac{\partial W}{\partial \lambda^a}, \quad (10)$$

$$\dot{\phi} = -\frac{1}{3} W. \quad (11)$$

In our case we set to zero all scalar fields except  $m$  and the  $SU(3)$  singlet in the decomposition (8), hereafter called  $\sigma$ . The SYM operator corresponding to that field is  $\text{Tr } \lambda^4 \lambda^4$ , i.e. the  $\mathcal{N} = 1$  gaugino condensate. In terms of these fields, the superpotential is [16]

$$W = \frac{3}{4} \left( \cosh \frac{2m}{\sqrt{3}} + \cosh 2\sigma \right). \quad (12)$$

The supergravity e.o.m. (11) can be solved exactly [16]:

$$\phi(y) = \frac{1}{2} \log[2 \sinh(y - C_1)] + \frac{1}{6} \log[2 \sinh(3y - C_2)], \quad (13)$$

$$m(y) = \frac{\sqrt{3}}{2} \log \left[ \frac{1 + e^{-(y-C_1)}}{1 - e^{-(y-C_1)}} \right], \quad (14)$$

$$\sigma(y) = \frac{1}{2} \log \left[ \frac{1 + e^{-(3y-C_2)}}{1 - e^{-(3y-C_2)}} \right]. \quad (15)$$

This solution is singular, but this singularity is acceptable, as long as  $C_1 \geq C_2/3$  [18]. Here and below we have rescaled the  $AdS_5$  radius to  $L = 1$ .

The asymptotic UV behavior of  $m(y)$  and  $\sigma(y)$  is

$$m(y) \sim \sqrt{3} e^{C_1} e^{-y}, \quad \sigma(y) \sim e^{C_2} e^{-3y}, \quad y \rightarrow \infty. \quad (16)$$

These equations show that  $m$  is a true deformation of  $\mathcal{N} = 4$  SYM, with UV scaling dimensions 3, and that  $\sigma$  is indeed the VEV  $\langle \text{Tr } \lambda^4 \lambda^4 \rangle$ .

The latter identification deserves an explanation, as it will be useful later.

A supergravity scalar  $\lambda$  of mass  $M$  behaves at large  $y$  as

$$\lambda(y) = \lambda_0 e^{(\Delta-4)y} + C e^{-\Delta y}. \quad (17)$$

In the holographic interpretation,  $\lambda_0$  is the source of a dimension- $\Delta$  operator,  $O$ , and the partition function  $Z[\lambda_0] = \langle \exp(-\int d^4x \lambda_0 O) \rangle$  is given by the supergravity action  $S[\lambda]$  computed at the stationary point with boundary condition  $\lim_{y \rightarrow \infty} \exp[(4 - \Delta)y] \lambda(y) = \lambda_0$  [3]:

$$\langle e^{-\int d^4x \lambda_0 O} \rangle = e^{-S[\lambda]} \Big|_{\text{stationary point}}. \quad (18)$$

Choosing for simplicity a canonical kinetic term for  $\lambda$ , and substituting eq. (17) into eq. (18) we immediately find:

$$\langle O \rangle = \frac{\delta S}{\delta \lambda_0} \Big|_{\lambda_0=0} = - \int dy \frac{\partial}{\partial y} \left[ e^{4\phi(y)} e^{(4-\Delta)y} \frac{\partial \lambda}{\partial y} \right] = \Delta C. \quad (19)$$

In the case of  $\sigma$ ,  $\Delta = 3$ , so that its asymptotic behavior –  $\sigma \sim \exp(-3y)$  – is the correct one to generate a gaugino condensate.

For further details we refer the interested reader to ref. [16], which also contains a detailed study of the Wilson loop in the background given in eqs. (13,14,15).

We move now to a brief discussion of the RG equations in the holographic framework.

Let us restore dimensions to  $y$  and introduce again the  $AdS_5$  radius  $L$ . The supergravity action is divergent on an asymptotically  $AdS$  background. To regularize it, one can excise the asymptotic region  $y > L \log(L\Lambda)$ ;  $\Lambda$  is clearly a UV cutoff. When all fields are independent of the 4-d coordinates, the supergravity action depends on the coordinate  $y$  only through the scale factor  $\phi(y)$ . We will find it useful to define a new coordinate  $\mu = \exp[\phi(y)]/L \leq \Lambda$ . In terms of this new coordinate, a generic 5-d metric can be written as

$$ds^2 = \omega(\mu)^2 d\mu^2 + \mu^2 g_{\mu\nu}(\mu, x) dx^\mu dx^\nu. \quad (20)$$

Here  $\omega(\mu) = \mu^{-1} dy/d\phi$  and the metric  $g_{\mu\nu}(\mu, x)$  is asymptotically flat:  $g_{\mu\nu}(\mu, x) \approx \eta_{\mu\nu}$  at large space-like  $x$ . A plausible definition of the holographic renormalization group is as follow (see also [19, 20, 21]). Define “bare” fields, independent of  $\mu$  as

$$\lambda_B^a(x) = \lambda^a(\mu = \Lambda, x), \quad g_{B\mu\nu}(x) = g_{\mu\nu}(\mu = \Lambda, x). \quad (21)$$

Define also  $\mu$ -dependent “renormalized” fields as the fields  $\lambda^a(\mu, x)$ ,  $g_{\mu\nu}(\mu, x)$  that solve the supergravity e.o.m. with boundary conditions  $\lambda_B^a(x)$ ,  $g_{B\mu\nu}(x)$ <sup>2</sup>. Clearly, the “bare” supergravity action is independent of  $\mu$ ; therefore,

$$0 = \mu \frac{d}{d\mu} S[\lambda_R^a, g_{B\mu\nu}, \mu] = \mu \frac{\partial}{\partial \mu} S + \int d^4x \left[ \frac{d\lambda_R^a}{d\mu} \frac{\delta S}{\delta \lambda_R^a}(x) + \frac{dg_{B\mu\nu}}{d\mu} \frac{\delta S}{\delta g_{B\mu\nu}}(x) \right]. \quad (22)$$

This equation is not a tautology once one gives independent equations for  $\dot{\lambda}_R^a \equiv d\lambda_R^a/d\mu$  and  $\dot{g}_{B\mu\nu} \equiv dg_{B\mu\nu}/d\mu$  (the beta functions). The beta function equations are obtained by splitting the supergravity action  $S$  into its UV and IR parts:

$$S = S_{UV}[\mu] + S_{IR}[\mu] = \int d^4x \int_\mu^\Lambda d\nu \mathcal{L}(\nu, x) + \int d^4x \int_{\mu_0}^\mu d\nu \mathcal{L}(\nu, x). \quad (23)$$

Here,  $\mu_0$  stands for the physical IR cutoff,  $\mu_0 = 0$  if the IR theory is conformal. Finally, the RG equations are

$$\dot{\lambda}_R^a(\mu, x) = -\frac{\delta S_{UV}}{\delta \lambda_R^a}(\mu, x), \quad \dot{g}_{B\mu\nu}(\mu, x) = -\frac{\delta S_{UV}}{\delta g_{B\mu\nu}}(\mu, x). \quad (24)$$

Eqs. (22,24) define a holographic renormalization scheme where the beta functions are  $\beta^a \equiv \dot{\lambda}_R^a(\mu, x)$ ,  $\beta_{\mu\nu} = \dot{g}_{B\mu\nu}(\mu, x)$ . Notice that because of the definition of the renormalized metric,  $\beta_{\mu\nu}$  vanishes on translationally-invariant backgrounds.

Eqs. (22,24) also suggest a natural candidate central function  $c(\mu)$ .

Let us briefly recall the properties of a central function.

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<sup>2</sup>To be precise, we also need appropriate boundary conditions at the IR boundary. In all concrete cases in the literature, it is not difficult to find them out explicitly.

1.  $c(\mu)$  must decrease along an RG trajectory:  $c_{IR} \equiv \lim_{\mu \rightarrow 0} c(\mu) \leq \lim_{\mu \rightarrow \infty} c(\mu) \equiv c_{UV}$ .
2.  $\dot{c}(\mu) = 0$  at the fixed points of the RG group (conformal points).
3. At the fixed points,  $c$  is one of the central charges of the conformal algebra. In CFT described by supergravity duals, i.e. at large 't Hooft coupling, there exists only one central charge [22]

$$\langle T^\mu_\mu(x) \rangle = c \left( -\frac{1}{8} R^{\mu\nu} R_{\mu\nu} + \frac{1}{24} R^2 \right). \quad (25)$$

A central function obeying all these properties was found in [6] (see also [17]):

$$c[\mu] = \text{const} \left( \dot{\phi} \right)^{-3}. \quad (26)$$

It is monotonic because of the following equation [23], that can be taken as the definition of the holographic scheme

$$\dot{c} = 2c \dot{\lambda}^a \dot{\lambda}^b G_{ab}. \quad (27)$$

This equation follows from the supergravity e.o.m. (5); here  $G_{ab}$  is the scalar kinetic term, not necessarily canonical. Monotonicity of  $c$  along a generic RG trajectory can also be proven using the null energy condition [17].

Our definition of  $c$  is unique only at the critical points  $\dot{c} = 0$ . Away from criticality,  $c$  need not coincide with central functions defined in other ways; indeed, it need not coincide with other holographic definitions of  $c$ , as for instance that of ref. [19]. This non-uniqueness, even within the holographic scheme, is due to the ambiguity in the identification of  $\phi$  as a function of the scale  $\mu$ . The standard identification  $\phi = \log(\mu/\mu_0)$  is unique only at the critical points  $\dot{c} = 0$ , because of the AdS/CFT correspondence. Away from criticality, uniqueness is lost.

A function that reduces to  $c$  at the RG fixed points can be defined in any field theory by computing the two point function of the stress-energy tensor using the equation [24]

$$\langle T_{\mu\nu}(x) T_{\rho\sigma}(0) \rangle = -\frac{1}{48\pi^4} \Pi_{\mu\nu\rho\sigma}^{(2)} \left[ \frac{c_H(x)}{x^4} \right] + \pi_{\mu\nu} \pi_{\rho\sigma} \left[ \frac{f(x)}{x^4} \right], \quad (28)$$

where  $\pi_{\mu\nu} = \partial_\mu \partial_\nu - \eta_{\mu\nu} \partial^2$ , and  $\Pi_{\mu\nu\rho\sigma}^{(2)} = 2\pi_{\mu\nu} \pi_{\rho\sigma} - 3(\pi_{\mu\rho} \pi_{\nu\sigma} + \pi_{\mu\sigma} \pi_{\nu\rho})$ . We call  $c_H$  the *canonical*  $c$ -function.

In a generic field theory,  $c_H(x)$  is not monotonic [24]. In theories admitting a supergravity dual it is, as we shall now see. The holographic correspondence eq. (18) extends straightforwardly to  $T_{\mu\nu}$  once we find the source that couples to the stress-energy tensor.

To do that, we expand the metric as  $g_{\mu\nu}(y, x) = \exp[2\phi(y)]\eta_{\mu\nu} + \delta g_{\mu\nu}$ . For  $y \rightarrow +\infty$  we have  $\delta g_{\mu\nu}(y, x) = \exp(2y/L)h_{\mu\nu}(x) + O(1)$  so that the source is  $h_{\mu\nu}$ . Eq. (18) now reads

$$\langle e^{-\int d^x T^{\mu\nu} h_{\mu\nu}} \rangle = e^{-S[g_{mn}]}, \quad (29)$$

where  $g_{55} = 1$ ,  $g_{\mu 5} = 0$ , and  $g_{\mu\nu}$  obeys the boundary condition

$$\lim_{y \rightarrow +\infty} e^{-2y/L} g_{\mu\nu}(y, x) = \eta_{\mu\nu} + h_{\mu\nu}(x). \quad (30)$$

To find the two-point function of  $T_{\mu\nu}$  we compute the supergravity action to quadratic order in  $h$

$$\langle T_{\mu\nu}(x) T_{\rho\sigma}(0) \rangle = \frac{\delta^2 S}{\delta h^{\mu\nu}(x) \delta h^{\rho\sigma}(0)} \Big|_{h=0}. \quad (31)$$

The on-shell supergravity action has the following form

$$S[h_{\mu\nu}] = \int d^4 x dy e^{4\phi(y)} \delta g_{\mu\nu}(y, x) \square^{-1} \Pi^{(2)\mu\nu\rho\sigma} \delta g_{\rho\sigma} + \dots, \quad (32)$$

Ellipsis denote terms proportional to the trace of the metric. Because of eq. (32), the transverse-traceless part in eq. (28) can be written as  $\square^{-2} \Pi_{\mu\nu\rho\sigma}^{(2)} G(x)$ , where  $G(x)$  is the boundary-to-boundary Green function of a 5-d minimally coupled massless scalar.

$G(x)$  can be computed as follows. Consider a minimally-coupled massless scalar propagating in the background eq. (3). It obeys the equation of motion

$$\left( \partial_y e^{4\phi} \partial_y + e^{2\phi} \square \right) \psi(y, x) = 0. \quad (33)$$

Its Fourier transform near the boundary is

$$\tilde{\psi}(y, k) = \left[ 1 + a_1 e^{-2y/L} k^2 + a_2 y e^{-4y/L} k^4 + e^{-4y/L} \tilde{G}(k) + o(e^{-4y/L}) \right] \tilde{\psi}(k); \quad (34)$$

$\tilde{G}(k)$  is the Fourier transform of  $G(x)$ .

A detailed calculation of  $G(x)$  and the canonical  $c_H(x)$  that results from it, in a few cases where the computation can be done analytically, has been reported elsewhere [23] (see also [25]). For the two flows examined in ref. [23] it was found that  $c_{HIR} \equiv \lim_{|x| \rightarrow 0} c_H(x) < c_{HUV} \equiv \lim_{|x| \rightarrow \infty} c_H(x)$ . As we mentioned above, this property is not generic in 4-d CFT; counterexamples were found in [24]. In theories with holographic supergravity duals, though,  $c_{HIR} \leq c_{HUV}$ . This inequality is obvious when  $c_{HIR} = 0$ , since positivity of the two-point function eq. (28) implies  $c_H(x) \geq 0$  [24].

When  $c_{HIR} > 0$ , the IR fixed point is a nontrivial CFT. In this case, the inequality  $c_{HIR} \leq c_{HUV}$  holds because at the fixed point  $c_H(x)$  and  $c(y)$  coincide [22], because  $c(y)$  is monotonic (eq. (27)), and because the usual UV/IR connection [1] holds near the fixed points:  $|x| \rightarrow \lambda|x| \sim y \rightarrow y - L \log \lambda$  for  $|y| \rightarrow \infty$ . The last fact is more or less obvious; but we can also prove the inequality quite easily as follows [21].



Let us define the quantity

$$\tilde{G}_y(k) \equiv e^{4\phi(y)} \frac{\partial_y \tilde{\psi}(y, k)}{\tilde{\psi}(y, k)}. \quad (35)$$

It obeys  $\lim_{y \rightarrow \infty} \tilde{G}_y(k) = \tilde{G}(k) + ak^2 + bk^2$ , where  $a$  and  $b$  are constants. Because of eq. (33),  $\tilde{G}_y(k)$  satisfies the equation [21]

$$\partial_y \tilde{G}_y(k) = k^2 e^{2\phi(y)} - e^{-4\phi(y)} [\tilde{G}_y(k)]^2. \quad (36)$$

Expanding  $\tilde{G}(k)$  near  $k^2 = 0$  we find  $\text{Im } \tilde{G}(k) = O(k^4)$  and  $\text{Re } \tilde{G}(k) = O(k^2)$ . Keeping only the lowest non-vanishing terms in  $k^2$  in eq. (36) and using the initial conditions given above we find

$$\text{Re } \tilde{G}_y(k) = O(k^2), \quad \partial_y \text{Im } \tilde{G}_y(k) = -2e^{-4\phi(y)} \text{Re } \tilde{G}_y(k) \text{Im } \tilde{G}_y(k) = O(k^2) \text{Im } \tilde{G}_y(k). \quad (37)$$

This equation implies that  $\text{Im } \tilde{G}_y(k) = \text{Im } \tilde{G}(k) + O(k^6)$ . Since  $\text{Im } \tilde{G}(k) = (4\pi)^{-1} c_{HIR} k^4 + O(k^6)$  and  $\lim_{y \rightarrow -\infty} \text{Im } \tilde{G}_y(k) = (4\pi)^{-1} c_{IR} k^4 + O(k^6)$  [2], we obtain  $c_{HIR} = c_{IR} \leq c_{UV} = c_{HUV}$ . The last equality is obvious.

Finally, let us give a holographic formulation of the Goldstone theorem. The key ingredient here is that global symmetries of the 4-d theory correspond to 5-d *gauge* symmetries of the supergravity dual [3]. The boundary value  $A_\mu(x)$  of the 5-d gauge field is the source of the Noether current associated to the symmetry. Let us call  $B$  the expectation value of an operator  $O$  in the presence of a 5-d gauge field:

$$B = B^0 + \int d^4 k B_\mu^1(-k) \tilde{A}^\mu(k) + O(\tilde{A}_\mu^2). \quad (38)$$

Here  $A_5 = 0$  by gauge choice. The relation between  $B$  and the asymptotic form of its associated 5-d field is given by eq. (19). Since  $A_\mu(x)$  is the source of the conserved current  $J_\mu(x)$ ,  $B_\mu^1(-k)$  is the two-point function  $\langle \tilde{J}_\mu(-k) O(0) \rangle$ .

A pure-gauge field,  $\tilde{A}_\mu = k_\mu \Lambda(k)$ , can be set to zero with a gauge transformation *that acts as the 4-d symmetry on  $O$* . We find then another expression for  $B$ :

$$B = B^0 + \int d^4 k \delta B(-k) \Lambda(k) + O(\Lambda^2), \quad \delta B(-k) \equiv \langle \delta_\Lambda O \rangle. \quad (39)$$

If  $\lim_{k \rightarrow 0} \delta B(-k) \equiv \delta B$  is nonzero, eqs. (38,39) imply  $\lim_{k \rightarrow 0} k^\mu B_\mu^1(-k) = \delta B \neq 0$ . By Lorentz invariance  $B_\mu^1(-k) = k_\mu B^1(-k)$  and  $B^1(-k) = \delta B/k^2$  for  $k \rightarrow 0$ . This means that the two-point function  $\langle \tilde{J}_\mu(-k) O(0) \rangle$  has a massless pole, physical since  $J_\mu$  and  $O$  are both gauge invariant. Notice that the only point where we used holography was in the identification of the source  $A_\mu$  with a 5-d gauge field.

In this note, we have surveyed various aspects of the holographic duality between strongly interacting 4-d field theories and 5-d supergravity, and we have found the holographic dual of several features of field theory. This “dictionary” allows for the study of strongly interacting theories by means of classical, weak-curvature (super)gravity.

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